Separable ODE $y' = F(x,y) \Leftrightarrow F(x,y) = g(x)h(y)$

1. Factor $F(x,y) = g(x)h(y)$
2. Separate into $\frac{dy}{dx} = g(x)h(y) \Rightarrow \frac{dy}{h(y)} = g(x)dx$
3. Integrate $\int \frac{dy}{h(y)} = \int g(x)dx$

Ex: $\sqrt{4-x^2}y' = \frac{1}{lny} \Rightarrow \int lny dy = \int \frac{dx}{\sqrt{4-x^2}}$

$\boldsymbol{ylny - y = \sin^{-1}\frac{x}{2} + C}$

Linear ODE $y' + P(x)y = f(x)$

1. Find integrating factor $I(x)$ or $\mu(x) = e^{\int P(x)dx}$
2. Solution is $I(x)y = \int I(x)f(x)dx$
3. Evaluate RHS

Ex: $x^2y + 3xy = 2\sin\frac{x}{3} \Rightarrow P(x) = \frac{3}{x}, f(x) = \frac{2}{x^2}\sin\frac{x}{3}, I(x) = e^{3\int\frac{dx}{x}}$

$I(x) = e^{3lnx} = e^{lnx^3} = x^3$

$\therefore \Rightarrow \boldsymbol{x^3y = \int 2x\sin\frac{x}{3}dx}$

Exact ODE Form $M(x,y)dx + N(x,y)dy = 0$

Exact $\Rightarrow \exists\, f(x,y) \ni Mdx + Ndy = df$ & $\Rightarrow$ LHS of ODE is total differential $u(x,y)$, $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$

1. Determine Exact or not
2. Solve $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \Rightarrow$ total differential of f
3. If Exact, $M(x,y) = \frac{\partial f}{\partial x}$ & $N(x,y) = \frac{\partial f}{\partial y} \Rightarrow M_y = N_x$
4. $f(x,y) = C \Rightarrow find\ f \quad \int df = f = C$
   $\int dy = 0 \quad y = C$
5. $M_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right),\ N_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$
6. $f(x,y) = \int M(x,y)dx = H(x,y) + g(y)$
   $f(x,y) = \int N(x,y)dy = G(x,y) + h(x)$

Ex: $(1+2x)ydy + (x - 2 + y^2)dx = 0,\ w/\ M = [1]$ & $N = [2]$

$N_x = 2y \quad M_y = 2y$

$\therefore \because N_x = M_y,\ ODE\ is\ Exact$

$[1]\int(y + 2xy)dy + [2]\int(x - 2 + y^2)dx$

$[1]\ M = \frac{y}{2} + xy^2 + h(x)$

$[2]\ N = \frac{x^2}{2} - 2x + xy^2 + g(y)$

$\boldsymbol{f(x,y) = \frac{y}{2} + xy^2 + \frac{x^2}{2} - 2x = C}$

Homogeneous ODE Form $m(x,y)dx + N(x,y)dy = 0$ is homogeneous $\Leftrightarrow$ both $M(x,y)$ & $N(x,y)$ are homogenous of same order**,** $f(x,y)$ is HG of order $\alpha \Leftrightarrow f(tx,ty) = t^\alpha f(x,y)$

1. Substitute $y = ux$ or $x = vy$ into ODE
2. Simplify into Separable or Linear
3. Solve

Ex: $xy' = x + y \quad M(tx,ty) = tx + ty = t(x+y) \quad N(tx,ty) = tx$

$\therefore M\ \&\ N\ are\ HG\ \alpha = 1 \quad y = ux$

$dy = udx + xdu \quad u = \frac{y}{x}xdy = (x+y)dx$

$x(udx + xdu) = (x + ux)dx \quad x^2du = xdx$

$\int du = \int\frac{dx}{x} \quad u = C + lnx \quad \frac{y}{x} = \boldsymbol{lnx + C}$

Bernoulli ODE Form $y' + P(x)y = f(x)y^n$

1. Substitute $u = y^{1-n}$ (linear in u)
2. Reduce BF to linear $u' + P(x)u = f(x)$
3. Solve

Ex: $y' + \frac{1}{x}y = 2xy^{-2} \quad u = y^3 \quad y = u^{\frac{1}{3}}$

$\frac{dy}{dx} = \frac{1}{3}u^{-\frac{2}{3}}\frac{du}{dx}$

$\frac{1}{3}u^{-\frac{2}{3}}\frac{du}{dx} + \frac{u^{1/3}}{x} = 2xu^{-\frac{2}{3}} \quad \frac{du}{dx} + \frac{3u^{-1/2}}{x}$

$= 6xu \quad e^{\int 3u^{-\frac{1}{2}}du} = e^{6\sqrt{u}}$

$\frac{du}{dx} + \frac{3}{x}u = 6x \quad I(x) = e^{\int\frac{3}{x}dx} = e^{3lnx}$

$= x^3 \quad u' + P(x)u = g(x)$

$x^3u = \int x^3 6xdx \quad \boldsymbol{x^3y^3 = \frac{6}{5}x^5 + C}$

Non-Homogeneous ODE (NH)

1. Find complementary solution y_c
2. Find particular solution y_p using variation of parameters or undetermined coefficients
3. General Solution $\equiv y = y_c + y_p$

If Wronskian $\begin{cases}\neq 0, functions\ are\ \boldsymbol{Lin\ Indep} \\ = 0, functions\ are\ \boldsymbol{Lin\ Dep}\end{cases}$

Linear Constant Coefficients (LCC)

1. Guess $y = e^{mx}$ (auxiliary equation)
2. Substitute into (H) to get $y_c \quad x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
3. Solve to get $m_1, m_2, \dots m_n$
4. Forms
   I. $m_1 \neq m_2 \in \mathbb{R} \quad (D > 0)$
      $y_1 = e^{m_1x}, y_2 = e^{m_2x}$
      $\Rightarrow y = C_1e^{m_1x} + C_2e^{m_2x}$
   II. $m_1 = m_2 \in \mathbb{R} \quad (D = 0)$
      $y = e^{mx}$ & $RO \Rightarrow y_2 = xe^{mx}$
      $\Rightarrow y = C_1e^{mx} + C_2xe^{mx}$
   III. $m = \alpha \pm i\beta \quad (D < 0)$
      $\Rightarrow y = (C_1\sin\beta x + C_2\cos\beta x)e^{\alpha x}$

Reduction of Order, given y_1 solves $y'' + by' + cy = 0$ with $y = uy$

Substitute into ODE to get u-ODE

Ex: $y^{(3)} + y'' - 12y' = 0 \quad y = e^{mx} \Rightarrow m^3 + m^2m = 0$

$m(m^2 + m - 12) = (m+4)(m-3)m$

$\Rightarrow m = 0, -4, 3$

$\boldsymbol{y = C_1 + C_2e^{-4x} + C_3e^{3x}}$

Cauchy-Euler Equation (C-E) where $x^m \neq 0$

Let $y = x^m$ and substitute into ODE characteristic equation

$y = x^m, y' = mx^{m-1}, y'' = m(m-1)x^{m-2}, \dots$

I. $m_1 \neq m_2 \in \mathbb{R} \Rightarrow y = C_1x^{m_1} + C_2x^{m_2}$
II. $m_1 = m_2 \in \mathbb{R} \Rightarrow y = C_1x^m + C_2x^m lnx$
III. $m_1 = \overline{m_2} = \alpha \pm i\beta \Rightarrow y = x^\alpha[A\sin(lnx^\beta) + B\cos(lnx^\beta)]$

Ex: $x^2y'' - 5xy' + 9y = 0 \qquad m(m-1) - 5m + 9 = 0$

$m^2 - 6m + 9 \quad (m-3)^2 \Rightarrow m = 3$

$\boldsymbol{y = C_1x^3 + C_2x^3lnx}$

Undetermined Coefficients (signs are personal preference)

1. Solve corresponding (H) to obtain y_c or y_H
2. "Guess" form of y_p according to $f(x)$ w/ const
3. Determine coefficients using partial fractions
4. General solution to (NH) is $y = y_c + y_p$

$f(x)$ can only be polynomial, exponential, sine, or cosine

$P_n(x) \Rightarrow y_p = a_nx^n + \cdots + a_1x + a_0$ w/o skipping terms

$e^{kx} \Rightarrow y_p = Ae^{kx}$

$\sin kx/\cos kx \Rightarrow y_p = A\sin kx + B\cos kx$

Form Ex: $f(x) = x^3 - 2 \Rightarrow y_p = Ax^3 + Bx^2 + Cx + D$

Ex: $y'' - 3y' - 4y = 3x - 7 \quad y_c = m^2 - 3m - 4 \quad (m-4)(m+1)$

$m = 4, 1 \Rightarrow y_c = C_1e^{4x} + C_2e^{-x} \quad y_p = Ax^2 + Bx + C$

$\boldsymbol{y = C_1e^{4x} + C_2e^{-x} + Ax^2 + Bx + C}$

Variation of Parameters

1. Find $y_c = \sum c_ny_n$
2. Find Wronskian
3. Find n #s of modified wronskians w/ n^th column $\equiv \begin{pmatrix}0\\ \vdots \\ F(x)\end{pmatrix}$
4. Evaluate $u_n = \int\frac{W_n}{W}dx$
5. Particular solution $y_p \equiv \sum u_ny_n$
6. General Solution $y = y_c + y_p$

Ex: $y''' + 9y' = \sec 3x \quad y''' + 9y' \Rightarrow m^3 + 9m = 0 \Rightarrow m = 0 \pm 3i$

$y_c = C_1 + C_2\cos 3x + C_3\sin 3x$

$W = \begin{Vmatrix}1 & \cos 3x & \sin 3x\\ 0 & -3\sin 3x & 3\cos 3x\\ 0 & -9\cos 3x & -9\sin 3x\end{Vmatrix}$

$= 1\begin{Vmatrix}-3\sin 3x & 3\cos 3x\\ -9\cos 3x & -9\sin 3x\end{Vmatrix} = 27$

$W_1 = \begin{Vmatrix}1 & \cos 3x & \sin 3x\\ 0 & -3\sin 3x & 3\cos 3x\\ 0 & -9\cos 3x & -9\sin 3x\end{Vmatrix} = 3\sec 3x$

$u_1 = \int\frac{W_1}{W}dx$

$= \frac{1}{9}\int\sec 3x\,dx = \frac{1}{9}ln|\sec 3x + \tan 3x|$

$\boldsymbol{y = y_c + y_p = C_1 + C_2\cos 3x + C_3\sin 3x}$

$\boldsymbol{+ \overbrace{\frac{1}{9}ln|\sec 3x + \tan 3x|}^{u_1y_1} + \overbrace{\frac{1}{9}xcos3x}^{u_2y_2} + \underbrace{\frac{1}{9}ln|\sec 3x|}_{u_3}\underbrace{\sin 3x}_{y_3}}$

LRC System

$\boldsymbol{L\ddot{q} + R\dot{q} + \frac{1}{C}q = E(t)}$ or $\boldsymbol{V(t)}$ w/ inductance, resistance, capacitance, current, and imposed voltage

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Mass-Spring System

$\boldsymbol{m\ddot{x} + \beta\dot{x} + kq = F(t)}$ w/ speed, displacement, spring constant, and damping constant

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$\boldsymbol{f(t) * g(t) \rightarrow}$

$\boldsymbol{\int_0^t f(t-u)g(u)du}$ **or** $\boldsymbol{\int_0^t f(u)g(t-u)du}$

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